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Solitons in Plasmas: A Lie Symmetry Approach

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Abstract This paper talks about the stationary solitons for Langmuir waves in plasmas that are described by the Nonlinear Schrödinger's equation with power law nonlinearity. The integration is carried out by the usage of Lie symmetry in presence of perturbation terms.

Keywords Solitons · Lie symmetry · Integrability

1 Introduction

The nonlinear Schröinger's equation (NLSE) in its dimensionless form has important applications in Plasma Physics [1–10]. It describes the electron (Langmuir) waves [2, 9]. The NLSE is given by

$$iq_t + aq_{xx} + b|q|^{2n}q = 0 \tag{1}$$

where a and b are constants and n is the index of power law. Equation (1) is not integrable by the method of Inverse Scattering Transform (IST) [6]. However the integration of (1) can be carried out using other means. They are travelling wave hypothesis or Lie symmetry approach and others [1]. The solutions of (1) are called *solitons*.

The emission of Langmuir waves in the form of small-scale localized electrostatic bursts have been observed directly in waveform data in many space plasma environments, such as solar wind, auroral region and polar cap. The comparison of observations in various space regions has shown that emissions are seen in associations with warm electron fluxes and have common characteristic properties such as burst-like character, an irregular structure,

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amplitude variations and a low frequency modulation. Langmuir wave bursts occur in association with electron fluxes with energies 100–400 eV propagating from distant regions of the magnetosphere during magnetic disturbances. The results of bicoherence analysis of wave data have shown that usually the parametric decay process do not play an important role in the formation of Langmuir wave bursts. It has been found that a typical power spectrum width of single burst is about 10% of the local plasma frequency, which is larger than the width generated by the thermal effect in Langmuir dispersion. Moreover, power spectra have usually a characteristic form with a dent in the upper part. Power time evolution studies show that these small-scale bursts tend to be correlated with the level of the low frequency wave power. Thus in the framework of the electron beam-plasma interaction, the presence of the low frequency turbulence is expected to play a prominent role in the generation of these plasma oscillations. The theoretical model in the quasi-linear statistical approximation has been developed for a beam-plasma instability in the magnetized plasma in the presence of low frequency turbulence. It has been shown that the beat-type waveforms of Langmuir emissions can be explained by interference between waves excited by an electron beam and scattering off the density fluctuations. The frequency width of the burst spectrum increases sufficiently due to the resonant wave scattering providing the wave power access to phase space regions with low growth rates [6, 7].

An important property of (1) is that it has at least three integrals of motion. They are given by

$$N=\int_{-\infty}^{\infty}|q|^2dx, \tag{2}$$

$$M=ia\int_{-\infty}^{\infty}(qq_x^*-q^*q_x)dx, \tag{3}$$

$$H=\int_{-\infty}^{\infty}\left(a|q_x|^2-\frac{b}{n+1}|q|^{2n+2}\right)dx \tag{4}$$

where N represents the plasmon number, while, M is the linear momentum of the soliton and H gives the Hamiltonian.

2 Perturbation Terms

The perturbed NLSE that is going to be considered in this paper is [1]:

$$iq_t+aq_{xx}+b|q|^{2n}q=\alpha|q|^2q_{xx}+\nu\frac{q_{xx}^*}{|q|^2}q^2 \tag{5}$$

where α and ν are constants. The α-term arises in the study of interaction between Langmuir waves and ion acoustic waves in plasmas, provided the velocity of the Langmuir waves is small as compared to the sound velocity [1]. The coefficient of ν accounts for the propagation of solitons in plasmas with sharp boundaries and dissipation. This term arises in the Gradov-Stenflo equation [1].

In this paper, the focus is going to be on obtaining the localized stationary solution to (5) of the form

$$q(x,t)=\phi(x)e^{-i\lambda t}, \tag{6}$$

where λ is a constant and the function ϕ depends on the variable x alone. Thus, from (5) and (6), $\phi(x)$ satisfies the time independent inhomogeneous nonlinear equation that is given by

$$\lambda\phi + a\phi'' + b\phi^{2n+1} = \alpha\phi^2\phi'' + \nu\phi'' \tag{7}$$

2.1 Mathematical Analysis

Equation (7) has a single lie point symmetry, namely $X = \partial/\partial x$. This symmetry will be used to integrate equation (7) once. It can be easily seen that the two invariants are

$$u = \phi \tag{8}$$

and

$$v = \phi' \tag{9}$$

Treating u as the independent variable and v as the dependent variable, (7) can be rewritten as

$$\frac{dv}{du} = \frac{\lambda u + bu^{2n+1}}{v(\alpha u^2 + \nu - a)} \tag{10}$$

Integrating (10) yields

$$\begin{aligned}\frac{v^2}{2} &= c_1 + \frac{\lambda}{2\alpha}\ln(\alpha u^2 + \nu - a)\\ &\frac{1}{2(n+1)(\nu - a)}\left\{bu^{2n+2}F\left(n+1, 1; n+2; -\frac{\alpha u^2}{\nu - a}\right)\right\}\end{aligned} \tag{11}$$

where c_1 is an arbitrary constant of integration and the Gauss' hypergeometric function is defined as

$$F(\alpha, \beta; \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)}\frac{z^n}{n!} \tag{12}$$

Now, reverting back to the variable ϕ gives

$$\begin{aligned}\left(\frac{d\phi}{dx}\right)^2 &= 2c_1 + \frac{\lambda}{\alpha}\ln(\alpha\phi^2 + \nu - a)\\ &\quad + \frac{1}{(n+1)(\nu - a)}\left\{b\phi^{2n+2}F\left(n+1, 1; n+2; -\frac{\alpha\phi^2}{\nu - a}\right)\right\}\end{aligned} \tag{13}$$

which leads to the quadrature

$$\begin{aligned}&x + c_2 =\\ &\int \frac{\sqrt{2\alpha(n+1)(\nu - a)}}{\sqrt{\alpha b\phi^{2n+2}F(n+1, 1; n+2; -\frac{\alpha\phi^2}{\nu - a}) + (n+1)(\nu - a)\{2\alpha c_1 + \ln(\alpha\phi^2 + \nu - a)\}}}d\phi\end{aligned} \tag{14}$$

where c_2 is an arbitrary constant of integration.

3 Conclusions

In this paper, the stationary soliton solution is obtained for relativistic plasmas. Here, the Lie symmetry approach is used to carry out the integration of the NLSE with power law nonlinearity. The resulting solution is in quadratures. In future, these results will be generalized to the case of NLSE with other laws of nonlinearity.

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